Phase transitions in scale-free neural networks: Departure from the standard mean-field universality class

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We investigate the nature of the phase transition from an ordered to a disordered state that occurs in a family of neural network models with noise. These models are closely related to the majority voter model, where a ferromagneticlike interaction between the elements prevails. Each member of the family is distinguished by the network topology, which is determined by the probability distribution of the number of incoming links. We show that for homogeneous random topologies, the phase transition belongs to the standard mean-field universality class, characterized by the order parameter exponent $\beta = 1/2$. However, for scale-free networks we obtain phase transition exponents ranging from 1/2 to infinity. Furthermore, we show the existence of a phase transition even for values of the scale-free exponent in the interval (1.5,2], where the average network connectivity diverges.

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I. INTRODUCTION

To determine the relationship between the topology of a complex network and its dynamical behavior has become a challenge in current scientific research. In the past few years there has been a great amount of work to find out the structural properties of complex networks, both theoretically and experimentally. These studies have revealed that a wide variety of networks occurring in nature exhibit nontrivial topologies [1–3]. Since the elements in many of these networks also interact dynamically, it is a fundamental problem to determine how the network topology affects its properties. Although some authors have recently addressed this problem in different contexts [4–16], the topic still remains largely unexplored.

Within this framework, ordering phenomena are particularly important because they provide us with general criteria for the existence or absence of collective behavior in multicomponent systems. Recent studies on ferromagneticlike networks have shown that the network topology dramatically affects their ordering processes. For example, in one-dimensional Ising models with small-world topology, there exists a phase transition from magnetized to demagnetized states even for very small values of the rewiring probability [4–10]. Another example is the propagation of an infectious disease in a population. When the network of infectious contacts has a homogeneous random topology, there exists a threshold for the fraction of vaccinated people above which the propagation stops. However, when such a network has a scale-free topology, the propagation never stops and the disease percolates across the entire population [11–14]. Results of this sort make it increasingly important to systematically analyze the relationship between the system's topology and its collective behavior.

In this paper we study the ordering processes that occur in complex neural networks with noise. The models we consider here are based on the McCulloch-Pits model of neuron-neuron interaction [17]. This model is closely related with the majority voter model, which has been considered for many years as a prototype for the study of ordering phenomena [18–22]. Our objective is to characterize the phase transitions from ordered to disordered states that occur when the noise parameter is changed. Using a mean-field theory approach, we find that when nontrivial topologies are implemented in the system, the universality class of this phase transition departs from the standard result in which the order parameter exponent is 1/2 [23]. It is worth mentioning that a somewhat similar departure was found in Ref. [24] for a phenomenological Landau theory of thermodynamic equilibrium systems. However, our results differ from the ones found by Goltsev *et al.* in several important aspects. First of all, we deal with a majority model (i.e., a genuine nonequilibrium model as described in Ref. [25]), on directed networks for which it is not clear that a thermodynamic formalism, through a phenomenological free energy, should hold. Second and more important, we assume scale-free topology only for the distribution of incoming connections, so the departure from the standard mean-field universality class arises from an entirely different mechanism than the one proposed in Ref. [24]. We will return to this point further on.

II. THE MODEL

The system we will be working with consists of a set of N discrete variables (voters) $\sigma_1, \sigma_2, \ldots, \sigma_N$, each acquiring only two possible values (opinions) $\sigma_n = +1$ or $\sigma_n = -1$, that change at each time step. The value of $\sigma_n(t+1)$ will be determined by another set of k_n voters according to the following scheme.

To each voter σ_n we assign a set $\mathcal{I}_n = \{\sigma_{i_1}(t), \sigma_{i_2}(t), \ldots, \sigma_{i_{k_n}}(t)\}$ of k_n different elements chosen randomly from anywhere in the system with equal probabil-

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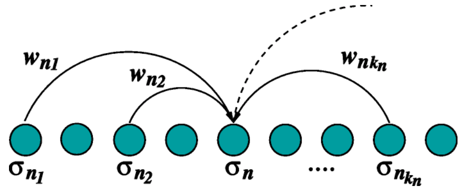

FIG. 1. Schematic representation of the model. The network consists of N elements (circles). Each element σ_n receives inputs from a set of other k_n elements randomly chosen from anywhere in the system. The connections are weighted with the factors w_{n_j}, so that the inputs do not all contribute equally to the state of σ_n.

ity (see Fig. 1). We will call $\mathcal{I}_n$ the set of *input elements* of σ_n.

We let k_n, the number of inputs of σ_n, be a random variable chosen from a probability distribution $P(k)$, which we will refer to as the *input probability distribution*. In this way the number of inputs may be different from one element to another. The probability function $P(k)$ determines the network topology. If $P(k)=e^{-K}K^k/k!$, we have the trivial homogeneous random topology in which every element has, on average, K inputs. We are interested here in the case $P(k) \sim k^{-\gamma}$, which corresponds to the so-called scale-free topology.

We define the majority function $M_n(t)$ associated with σ_n at time t as

$$M_n(t) = \mathrm{sgn}\left[\sum_{j=1}^{k_n} w_{n_j}\sigma_{n_j}(t)\right], \tag{1}$$

where w_{n_j} is the weight of the connection between σ_n and σ_{n_j}. These weights just indicate that the input elements σ_{n_j} might not all equally contribute to the majority function M_n. Different choices of the connection weights lead to different models of neural networks [17]. In this work we take the w_{n_j}'s as random variables distributed with probability $P_w(w)$.

Once every element σ_n of the network has been assigned with a set of input elements and a set of connection weights, the dynamical evolution of the network is given by

$$\sigma_n(t+1) = \begin{cases} M_n(t) & \text{with probability } 1-\eta, \\ -M_n(t) & \text{with probability } \eta. \end{cases} \tag{2}$$

The *noise intensity* η is the probability for the majority function to be violated. It is worth mentioning that when $\eta \neq 0$ there are no absorbing states in the system. In the context of the majority voter model, this parameter represents the "free will" of the voters; namely, it allows the possibility that even if the majority of my friends have one opinion about an issue, with probability η I can have the opposite opinion. The noise intensity η would be an analog to the temperature in ferromagnetic systems. The dynamics of the network can thus be set from purely deterministic to purely random by changing the value of η from 0 to 1/2. (Higher values of η would describe a system where individuals would tend to be of an opposite mind to the majority of the opinions that influence them; as we are updating all the σ_n variables simultaneously, such anarchism leads to essentially the same phenomenology as for $\eta < 1/2$ and will not be considered in this work.)

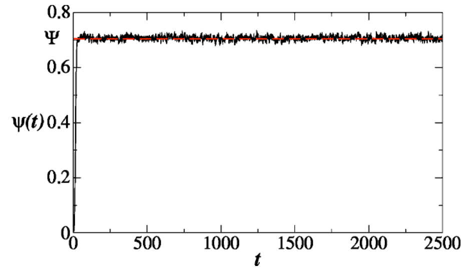

FIG. 2. (Color online) Temporal evolution of the order parameter $\psi(t)$. This graph was generated for a scale-free network with $N=20\,000$ elements, $\gamma=2$, $\eta=0.05$, and $w_{n_j}=1$ for every n_{i_j}. The stationary value Ψ of the order parameter is indicated by the dashed (colored) line.

Note that neither the set of input elements nor the connection weights change in time (quenched dynamics). Once they have been assigned to every element in the network, they remain fixed during the temporal evolution of the system.

III. PHASE TRANSITIONS

To characterize the ordering processes in the network, we introduce the instantaneous order parameter $\psi(t)$ as

$$\psi(t) = \lim_{N\to\infty} \frac{1}{N}\left\langle \sum_{n=1}^{N} \sigma_n(t) \right\rangle, \tag{3}$$

where $\langle \cdots \rangle$ denotes an average over the realizations of the system. This quantity is a measure of the order in the network: if the σ_n are randomly oriented then $\psi=0$, and $|\psi|=1$ if all are "aligned." In the limit $t\to\infty$, $\psi(t)$ reaches a stationary value Ψ (see Fig. 2).

We want to find Ψ as a function of the noise intensity η for arbitrary network topologies given by $P(k)$. The case $P(k)=\delta_{k,K}$ was analyzed in Ref. [23]. This case corresponds to a trivial homogeneous random topology in which every element has the same number K of inputs. It was shown that, for this trivial topology, there exists a critical value η_c of the noise intensity for which $\Psi \neq 0$ if $\eta < \eta_c$ and $\Psi = 0$ if $\eta \geq \eta_c$. Near the phase transition, Ψ behaves as $\Psi \approx (\eta_c - \eta)^{\beta}$ where $\beta = 1/2$. Therefore, the phase transition in this case belongs to what we will refer to as the *standard mean-field universality class*. As we show below, this is always true if all the moments of $P(k)$ are finite. However, the scale-free distribution $P(k) \sim k^{-\gamma}$ does not satisfy the above condition.

Figure 3 shows Ψ as a function of η for scale-free networks with $N=10^5$ elements and different values of the scale-free exponent γ. To obtain these curves we used the input probability distribution $P(k)=k^{-(\gamma-1)}-(k+1)^{-(\gamma-1)}$, which certainly behaves as $P(k)\sim(\gamma-1)k^{-\gamma}$ for large values of k [26]. Additionally, the connection weight distribution $P_w(w)=\delta(w-1)$ was used for these simulations. Similar results are obtained for other choices of $P_w(w)$ as long as it is not a symmetric function. It is apparent from Fig. 3 that a phase transition exists. However, its universality class de-

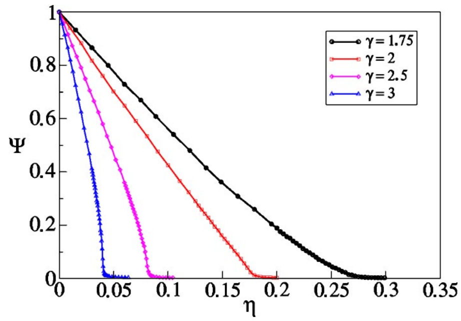

FIG. 3. (Color online) The order parameter Ψ as a function of the noise intensity η for different values of the scale-free exponent γ. These curves were numerically computed for scale-free networks with $N=10^5$ elements, $w_{n_j}=1$ for every n_j, and input probability distribution $P(k)=k^{-(\gamma-1)}-(k+1)^{-(\gamma-1)}$.

pends on the value of γ. In other words, near the phase transition Ψ behaves as $\Psi\sim(\eta_c-\eta)^{\beta}$, where now β is not necessarily $1/2$ but is a function of γ. It is important to note that the phase transition exists even for $\gamma=1.75$ and $\gamma=2$, which correspond to networks with infinite average connectivity. We will show that phase transitions will occur for any $\gamma>1.5$.

In order to find Ψ as a function of η, let $\phi_n(t)$ be the probability that $\sigma_n(t)=1$. The instantaneous order parameter $\psi(t)$ is then given by

$$\psi(t)=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}[2\phi_n(t)-1]. \tag{4}$$

Analogously, let $P_n^+(t)$ be the probability that $M_n(t)=1$ given that σ_n has k_n inputs. Note that $P_n^+(t)$ is just the probability for the sum $\sum_{j=1}^{k_n}w_{n_j}\sigma_{n_j}(t)$ to be positive. From the above definitions and Eq. (2), it follows that $\phi_n(t)$ obeys the dynamical equation

$$\phi_n(t+1)=\sum_{k_n=1}^{\infty}P(k_n)\{(1-\eta)P_n^+(t)+\eta[1-P_n^+(t)]\}. \tag{5}$$

The first term on the right-hand side of the above equation is the probability that the majority $M_n(t)$ at time t is positive and obeyed. The second term gives the probability that the majority at time t is negative but is violated. In both cases, σ_n will be positive at time $t+1$. It is important to note that $P_n^+(t)$ is a function of $\phi_{n_1}(t)$, $\phi_{n_2}(t)$, …, $\phi_{n_{k_n}}(t)$. Each $\phi_{n_j}(t)$ obeys an equation identical to Eq. (5), except for the subscripts. Therefore, Eq. (5) represents a hierarchy of nested nonlinear equations for the $\phi_n(t)$'s whose exact analytical treatment is out of our reach. Nonetheless, we can adopt a mean-field approach to solve this system of equations.

IV. MEAN-FIELD THEORY FOR P_n^+

The mean-field approach consists in the following assumptions: (1) All the σ_n's are statistically independent; (2) the probability $\phi_n(t)$ is the same for all the elements of the network and reaches a constant value in the stationary state: $\lim_{t\to\infty}\phi_n(t)=\phi$. These assumptions are valid as long as the k_n inputs of each σ_n are randomly selected from anywhere in the system with uniform probability. In doing so, the probability that two different elements share the same inputs is negligible for very large systems. Therefore, in the limit $N\to\infty$ one has $\mathcal{I}_n\cap\mathcal{I}_j=\emptyset$ for all $n\neq j$, which implies the statistical independence of the network elements. If we further assume that σ_n is statistically equivalent to any other σ_j, then the stationary value ϕ reached by the $\phi_n(t)$'s is site independent.

From the previous assumptions it follows that the stationary value of $P_n^+(t)$ is a function only of ϕ and k_n: $\lim_{t\to\infty}P_n^+(t)=P^+(\phi,k_n)$. Thus, in the limit $t\to\infty$, Eq. (5) becomes

$$\phi=\sum_{k_n=1}^{\infty}P(k_n)[(1-2\eta)P^+(\phi,k_n)+\eta]. \tag{6}$$

We can rewrite the above equation in terms of the stationary value Ψ of the order parameter by noting from Eq. (4) that $\Psi=2\phi-1$. Substituting $\phi=(\Psi+1)/2$ into Eq. (6) we obtain the following fixed point equation for Ψ:

$$\Psi=-1+2\sum_{k_n=1}^{\infty}P(k_n)[(1-2\eta)P^+(\Psi,k_n)+\eta]. \tag{7}$$

An exact expression for $P^+(\Psi,k)$ was found in Ref. [23] using the mean-field approach. For the sake of completeness we reproduce in the Appendix the derivation of this expression, which is

$$P^+(\Psi,k)=\frac{1}{2}+\sum_{m=1}^{k}a(k,m)\Psi^m, \tag{8}$$

where the coefficients $a(k,m)$ are defined as

$$a(k,m)=\frac{-i^{m+1}}{2\pi}\binom{k}{m}\int_{-\infty}^{\infty}[\hat{g}(\lambda)]^{k-m}[\hat{h}(\lambda)]^m\frac{d\lambda}{\lambda}. \tag{9}$$

In the last expression, $\hat{g}(\lambda)$ and $\hat{h}(\lambda)$ are the real and imaginary parts, respectively, of the Fourier transform of the connection weight distribution $P_w(w)$. From Eq. (9) and the fact that $\hat{h}(-\lambda)=-\hat{h}(\lambda)$ for any $P_w(w)$, it follows that $a(k,m)=0$ for even values of m. Therefore, only odd powers of Ψ contribute to the sum in Eq. (8). Additionally, if $P_w(w)$ is a symmetric function, then $\hat{h}(\lambda)=0$, which gives $P^+(\Psi,k)=1/2$. Consequently, for symmetric connection weight distributions $\Psi=0$ and there is no order in the system for any value of the noise intensity η.

Substituting the result (8) into Eq. (7) we get

$$\Psi=2(1-2\eta)\sum_{k=1}^{\infty}P(k)\sum_{m=1}^{k}a(k,m)\Psi^m. \tag{10}$$

In order to determine the existence and nature of a phase transition in the system we need to expand the right-hand side of this equation in a series of growing powers of Ψ.

V. SYSTEMS WITH HOMOGENEOUS RANDOM TOPOLOGY

A first approach to writing the right-hand side of Eq. (10) as a power series in Ψ is to interchange the two sums appearing in this equation. Assuming that this is possible Eq. (10) becomes

$$\Psi = 2(1-2\eta)\sum_{m=1}^{\infty} b_m \Psi^m, \tag{11}$$

where the coefficients b_m are given by

$$b_m = \sum_{k=m}^{\infty} P(k)a(k,m). \tag{12}$$

Near the phase transition $|\Psi| \ll 1$ and we can keep only the first two terms of the sum in Eq. (11), which gives

$$\Psi \approx 2(1-2\eta)(b_1\Psi + b_3\Psi^3). \tag{13}$$

In addition to the trivial solution $\Psi=0$, the above equation has two other nontrivial solutions given by

$$\Psi^2 \approx \frac{1}{|b_3|}\left(b_1 - \frac{1}{2(1-2\eta)}\right). \tag{14}$$

The critical value η_c of the noise at which the phase transition occurs is determined from the above equation by the condition $\Psi^2=0$, which gives

$$\eta_c = \frac{1}{2}\left(1 - \frac{1}{2b_1}\right). \tag{15}$$

On the other hand, it follows from Eq. (14) that, very close to the phase transition, Ψ behaves as

$$\Psi = \begin{cases} \dfrac{2b_1}{\sqrt{|b_3|}}(\eta_c - \eta)^{1/2} & \text{for } 0 < \eta_c - \eta \ll 1, \\ 0 & \text{for } \eta \geq \eta_c. \end{cases} \tag{16}$$

The above analysis shows that if the two sums in Eq. (10) can be interchanged, then the phase transition belongs to the standard mean-field universality class, for which the order parameter exponent is $\beta=1/2$. As we show next, such interchange is possible only if all the moments of $P(k)$ are finite, as for a Poisson distribution. However, this condition is not true for $P(k) \sim k^{-\gamma}$.

VI. THE CENTRAL LIMIT THEOREM

Although Eq. (9) gives the exact value of $a(k,m)$ for a particular connection weight distribution $P_w(w)$, the central limit theorem implies that $a(k,m) \sim k^{m/2}$ for large values of k and any $P_w(w)$ with finite first and second moments.

As mentioned above, $P^+(\Psi,k)$ is the probability that the sum $\xi_k = \sum_{j=1}^{k} w_{n_j}\sigma_{n_j}$ appearing in the definition of the majority rule, evaluates to a positive number. The k terms in this sum are independent identically distributed random variables with mean $\mu = \Psi\langle w\rangle$ and variance $\Delta^2 = \langle w^2\rangle - \Psi^2\langle w\rangle^2$, where $\langle w\rangle$ and $\langle w^2\rangle$ are the first and second moments of $P_w(w)$, respectively. Therefore, the central limit theorem applies to the distribution of ξ_k for large values of k, for which $P^+(\Psi,k)$ will be given by

$$P^+(\Psi,k) \approx \frac{1}{\sqrt{2\pi k\Delta^2}}\int_0^{\infty} \exp\left[-\frac{(\xi - k\mu)^2}{2k\Delta^2}\right] d\xi \tag{17}$$

$$= \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{\mu k^{1/2}}{\sqrt{2}\Delta}\right)\right] \tag{18}$$

$$= \frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{\pi}(2n+1)n!}\left(\frac{\mu k^{1/2}}{\sqrt{2}\Delta}\right)^{2n+1}. \tag{19}$$

Comparing the last expression with Eq. (8) (making $m = 2n+1$ and $\Psi = \mu/\langle w\rangle$), we obtain that $a(k,m) \sim k^{m/2}$ for large values of k. This result prevents us from changing the order of summation in Eq. (10) for an arbitrary input distribution $P(k)$. Indeed, since $a(k,m) \sim k^{m/2}$ for large k, the coefficients b_m appearing in Eq. (11) are well defined only if all the moments of $P(k)$ are finite. However, when $P(k) \sim k^{-\gamma}$, the coefficients b_m diverge for $m \geq 2(\gamma - 1)$. Consequently, in this case the series in growing powers of Ψ of Eq. (10) cannot be obtained by interchanging the sums and the analysis leading to Eq. (16) is no longer valid.

VII. DEPARTURE FROM THE STANDARD MEAN-FIELD UNIVERSALITY CLASS

The results of the preceding sections show that if the moments of $P(k)$ are all finite, the phase transition predicted by Eq. (7) belongs to the standard mean-field universality class. This is not necessarily true for a scale-free distribution. Nevertheless, the existence and the nature of a phase transition in this case will still depend on the small Ψ expansion of the right-hand side of Eq. (7), which vanishes as $\Psi \to 0$, but is no longer analytic at $\Psi=0$. In order to evaluate the behavior for a scale-free distribution, it is necessary to extract first the singular part of Eq. (7). To this end, we begin by assuming that asymptotically $P(k) \sim k^{-\gamma}$, with $3/2 < \gamma < 5/2$. This ensures that $b_1 = \sum_{k=1}^{\infty} P(k)a(k,1)$ converges but $b_m = \sum_{k=m}^{\infty} P(k)a(k,m)$ diverges for $m > 1$. Then we consider the obvious identity

$$P^+(\Psi,k) = P^+(0,k) + \frac{\partial P^+(0,k)}{\partial \Psi}\Psi + \int_0^{\Psi}\int_0^{\psi'} \frac{\partial^2 P^+(\psi'',k)}{\partial \psi''^2} d\psi'' d\psi', \tag{20}$$

which in this case is useful because for $3/2 < \gamma < 5/2$, all the derivatives $\partial^m P^+(\Psi,k)/\partial \Psi^m$ diverge for $m \geq 2$ when $\psi \to 0$. Thus, the nonanalytic behavior of $P^+(\Psi,k)$ is contained in the last term of Eq. (20). From Eq. (8), which gives the exact value of $P^+(\Psi,k)$, we obtain $P^+(0,k) = 1/2$ and $\partial P^+(0,k)/\partial \Psi = a(k,1)$. Therefore, Eq. (20) becomes

$$P^{+}(\Psi,k)=\frac{1}{2}+a(k,1)\Psi+\int_0^{\Psi}\int_0^{\psi'}\frac{\partial^2 P^{+}(\psi'',k)}{\partial \psi''^2}d\psi'' d\psi' . \tag{21}$$

Substituting the last result into Eq. (7), and taking into account that $b_1=\Sigma_{k=1}^{\infty}P(k)a(k,1)$ [see Eq. (12)], we get

$$\Psi = 2(1-2\eta)\left[b_1\Psi+\sum_{k=1}^{\infty}P(k)\int_0^{\Psi}\int_0^{\psi'}\frac{\partial^2 P^{+}(\psi'',k)}{\partial^2\psi''}d\psi'' d\psi'\right]. \tag{22}$$

In the range $\gamma\in(3/2,5/2)$ that we are considering, the nonanalyticity comes from the large k behavior of the last term of Eq. (22). Thus, to determine its nonanalytic behavior, we are entitled to use the approximations to $P^{+}(\Psi,k)$ arising from the central limit theorem, in particular Eq. (18). In the small Ψ limit, $P^{+}(\Psi,k)$ can be written as

$$P^{+}(\Psi,k)\approx\frac{1}{2}\left[1+\mathrm{erf}\left(\frac{\Psi\langle w\rangle k^{1/2}}{\sqrt{2\langle w^2\rangle}}\right)\right]. \tag{23}$$

Substituting the last expression into Eq. (22) we get

$$\Psi\approx 2(1-2\eta)\left[b_1\Psi-\frac{1}{\sqrt{2\pi}}\left(\frac{\langle w\rangle^2}{\langle w^2\rangle}\right)^{3/2}\times\int_0^{\Psi}\int_0^{\psi'}\psi''\left(\sum_{k=1}^{\infty}P(k)k^{3/2}e^{-k\langle w\rangle^2\psi''^2/2\langle w^2\rangle}\right)d\psi'' d\psi'\right]. \tag{24}$$

The singular behavior of the sum can be evaluated by writing $P(k)=Ck^{-\gamma}$ (where C is a numerical constant) and replacing the sum by an integral over k. This procedure gives

$$\Psi\approx 2(1-2\eta)\times\left[b_1\Psi-\frac{2C}{\sqrt{\pi}}\left(\frac{\langle w\rangle^2}{2\langle w^2\rangle}\right)^{\gamma-1}\frac{\Gamma(5/2-\gamma)}{(2\gamma-3)(2\gamma-2)}\Psi|\Psi|^{2\gamma-3}\right], \tag{25}$$

which substitutes for Eq. (13) as the small Ψ approximation of the fixed point equation for scale-free topologies with $3/2<\gamma<5/2$.

For $\gamma=5/2$ a logarithmic correction appears and Eq. (25) should be replaced by

$$\Psi\approx 2(1-2\eta)\left[b_1\Psi-\frac{2C}{3\sqrt{\pi}}\left(\frac{\langle w\rangle^2}{2\langle w^2\rangle}\right)^{3/2}\Psi^3|\ln\Psi|\right]. \tag{26}$$

For $\gamma>5/2$, the coefficient b_3 converges and the behavior of the fixed point equation near the transition is accurately reproduced by Eq. (13). The nonanalytic terms appear as higher order corrections. In this case, the phase transition will fall again in the standard mean-field universality class as described in Eq. (14). On the other hand, for lower values of γ, namely, for $1<\gamma\leqslant 3/2$, the coefficient b_1 diverges. An analysis along the same lines as above shows that, in this case, the nonanalytic term appears as a power smaller than linear. Consequently, there is no phase transition: the system remains ordered for all $\eta<1/2$. However, the order parameter vanishes as $\Psi\sim(1/2-\eta)^{1/(3-2\gamma)}$ when $\eta\to 1/2$.

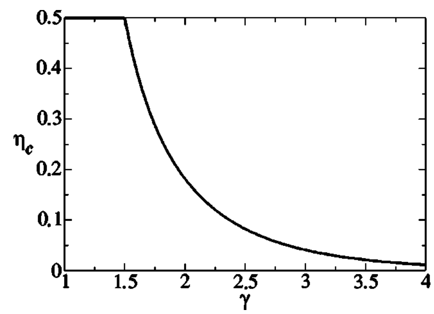

FIG. 4. Critical value η_c of the noise intensity as a function of the scale-free exponent γ. The curve is the graph of the analytic solution Eq. (15). Note that $\eta_c<1/2$ even for $3/2<\gamma\leqslant 2$, where the average network connectivity is infinite.

Thus, the range of interest is $3/2<\gamma\leqslant 5/2$ for which the behavior of the fixed point equation is described by Eqs. (25) and (26). In this range the system will exhibit a phase transition at a critical value of the noise intensity $\eta_c<1/2$. From Eqs. (25) and (26) we obtain $\eta_c=\frac{1}{2}(1-1/2b_1)$, just as in the standard mean-field case [see Eq. (15) and Fig. 4]. However, the phase transition in this case will not be characterized by the standard mean-field exponents. Indeed, for $3/2<\gamma<5/2$, Eq. (25) predicts that very close to the phase transition, Ψ behaves as

$$|\Psi|\sim\left[\frac{\sqrt{4\pi}(2\gamma-3)(2\gamma-2)b_1^2}{C\Gamma(5/2-\gamma)}\times\left(\frac{2\langle w^2\rangle}{\langle w\rangle^2}\right)^{\gamma-1}(\eta_c-\eta)\right]^{1/(2\gamma-3)}. \tag{27}$$

Thus, the exponent $\beta=(2\gamma-3)^{-1}$ characterizing the behavior of the order parameter near η_c is in the range $(\infty,1/2)$ as γ varies from $3/2^{+}$ to $5/2^{-}$. On the other hand, for $\gamma=5/2$, the logarithmic correction still gives a deviation from standard mean-field behavior. In this case, Eq. (26) gives

$$\Psi^2|\ln\Psi|\sim\frac{6\sqrt{\pi}b_1^2}{C}\left(\frac{2\langle w^2\rangle}{\langle w\rangle^2}\right)^{3/2}(\eta_c-\eta). \tag{28}$$

Figure 5 shows Ψ as a function of $\eta_c-\eta$ for different values of γ in a log-log plot. The symbols are the same numerical data as in Fig. 3 and the lines are the graphs of the theoretical results given in Eqs. (16), (27), and (28), depending upon the corresponding value of γ. We set $C=\gamma-1$, which is the normalization constant of the power-law distribution $P(k)=Ck^{-\gamma}$ when k varies continuously between 1 and ∞. We can see from Fig. 5 the excellent agreement between simulation and theory.

VIII. SUMMARY AND DISCUSSION

We have considered the phase transitions from ordered to disordered states that occur in a family of neural network models with noise. The existence and nature of the phase

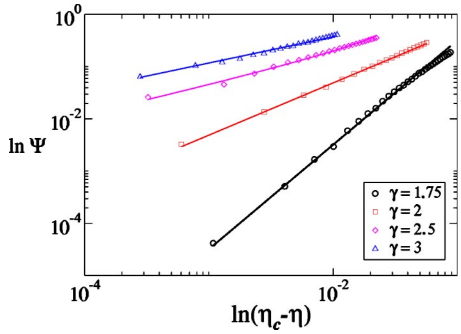

FIG. 5. (Color online) Log-log plot of Ψ as a function of $\eta_c - \eta$ for different values of the scale-free exponent γ. The symbols are the result of the simulations (same data as in Fig. 3) and the solid lines are the graphs of the theoretical solutions given in the text: Eq. (16) for $\gamma=3$, Eq. (28) for $\gamma=2.5$, and Eq. (27) for $\gamma=2$ and $\gamma=1.75$.

transition are strongly determined by the input probability distribution $P(k)$. If all the moments of $P(k)$ are finite (homogeneous random topologies), the phase transition belongs to the standard mean-field universality class; namely, in the limit $\eta \to \eta_c$ the order parameter behaves as $\Psi \sim (\eta_c - \eta)^\beta$ with $\beta = 1/2$.

Departure from the standard mean-field universality class might occur if $P(k)$ has divergent moments, as for scale-free networks where $P(k) \sim k^{-\gamma}$. In this case, the phase-transition exponent β is a function of the scale-free exponent γ. Table I summarizes how β and the critical noise η_c depend on γ. There are several important aspects to mention. First, for $\gamma \in (1, 3/2]$ the critical value of the noise is $\eta_c = 1/2$. This is the maximum value that the noise intensity can acquire and would correspond to infinite temperature in ferromagnetic-like systems. Therefore, in this case the network always exhibits long range order for any "finite temperature." The order parameter vanishes as $\Psi \sim (1/2 - \eta)^{1/(3-2\gamma)}$ when $\eta \to 1/2^-$.

Second, for $\gamma \in (3/2, 5/2)$ we have $\eta_c < 1/2$ and $\Psi \sim (\eta_c - \eta)^{1/(2-3\gamma)}$ when $\eta \to \eta_c^-$. This behavior of the order parameter represents a clear departure from the standard mean-field universality class. Although Goltsev *et al.* also found a similar result in a phenomenological Landau theory of ferromagnetic systems [24], their departure obeys an entirely different reason from the one we have found here. In

TABLE I. β and η_c for different values of γ. In the range $1 < \gamma \leq 3/2$ there is no phase transition since $\eta_c = 1/2$ is the maximum value that the noise intensity can acquire. For $\gamma = 5/2$ there is a logarithmic correction to $\beta = 1/2$.

$\Psi \sim (\eta_c - \eta)^\beta$		
	β	η_c
$1 < \gamma \leq 3/2$	$(3-2\gamma)^{-1}$	$1/2$
$3/2 < \gamma < 5/2$	$(2-3\gamma)^{-1}$	$\frac{1}{2}(1-1/2b_1)$
$\gamma = 5/2$	$1/2 + \log$	$\frac{1}{2}(1-1/2b_1)$
$5/2 < \gamma$	$1/2$	$\frac{1}{2}(1-1/2b_1)$

the networks we have considered, the scale-free topology is implemented only in the network of *input* connections. Since the input elements are chosen with uniform probability from anywhere in the system, it turns out that the *output* connections follow a Poisson distribution. Therefore, we never have isolated "hubs" dominating the dynamical behavior of the entire system, which is what causes the departure from standard mean-field behavior in Goltsev's work.

Third and probably more important, we have shown the existence of a phase transition even in the range $3/2 < \gamma \leq 2$, in which the average network connectivity is infinite. To our knowledge, this is the first example of a phase transition at "finite temperature" occurring in a network with infinite connectivity. This result is in contrast with the one reported by Goltsev *et al.*, where the first moment of $P(k)$ must be finite for the phase transition to exist.

In the majority voter networks we have analyzed, the phase transition will exist as long as the $\langle k^{1/2} \rangle$ moment of $P(k)$ is finite. This result can be understood by considering the quantity $\xi_k = \sum_{j=1}^k w_{n_j} \sigma_{n_j}$, appearing in the definition of the majority rule, as a random walk of k steps with average value $\mu_k = \langle w \rangle \Psi k$ and rms $\Delta_k = \sqrt{k(\langle w^2 \rangle - \Psi \langle w \rangle^2)}$. According to the central limit theorem (see Sec. VI), the convergence of the sum $\sum_k (\mu_k / \Delta_k) P(k) \sim \sum_k k^{1/2} P(k)$ is a necessary condition for the existence of the phase transition. This convergence guarantees that the quantity

$$\left\langle \frac{\mu_k}{\Delta_k} \right\rangle = \sum_{k=1}^{\infty} \frac{\mu_k}{\Delta_k} P(k)$$

remains finite. In such a case, there will always exist a "finite temperature" (namely, a critical value of the noise $\eta_c < 1/2$) at which the order in the system is destroyed.

For $\gamma > 5/2$ the phase transition falls again into the standard mean-field universality class. It is worth mentioning that some preliminary numerical results indicate that this is also the case for one-dimensional small-world networks with the same dynamical interaction we have presented here. The mean-field assumptions are not valid in this case since the elements are no longer statistically independent. Nevertheless, the phase transition in such networks is still in the standard mean-field class for any nonzero value of the rewiring probability. Whether the convergence of all the moments of $P(k)$ guarantees or not the standard mean-field universality class in this family of networks, regardless of the statistical independence of the elements, is still an open problem.

Finally, regarding the temporal evolution of the order parameter towards its steady value, a stability analysis of the evolution equation predicts that the order parameter ultimately decays exponentially to its stationary value except at the critical noise, where a power law decay ensues. For $3/2 < \gamma \leq 5/2$ the relaxation time characterizing the exponential decay behaves as $\tau \sim [(2\gamma - 3)(\eta_c - \eta)]^{-1}$, whereas at criticality $\Psi(t) \sim t^{-1/(2\gamma-3)}$. Numerical verification of these results as well as of the effect of a time varying threshold in Eq. (1) are currently under way.

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APPENDIX: COMPUTATION OF $P^{+}(\phi,k)$

Let us denote as $Q_k(\xi)$ the probability distribution function (PDF) of the quantity $\xi_k=[\Sigma_{j=1}^{k} w_{n_j}\sigma_{n_j}]$ appearing in the definition of the majority function, Eq. (1). The probability $P^{+}(\phi,k)$ that $\xi_k>0$ is then given by

$$P^{+}(\phi,k)=\int_0^{\infty} Q_k(\xi)d\xi. \qquad \text{(A1)}$$

The connection weights w_{n_j} are random variables chosen with probability $P_w(w)$, whereas in the stationary state, each σ_n acquires the values +1 and −1 with probabilities ϕ and $1-\phi$, respectively. Therefore, the PDF of the product $w_{n_j}\sigma_{n_j}$ is given by

$$P_{w\sigma}(x)=\phi P_w(x)+(1-\phi)P_w(-x). \qquad \text{(A2)}$$

Under the mean-field assumptions listed in Sec. IV, each term $w_{n_j}\sigma_{n_j}$ appearing in the sum ξ_k is an independent random variable with probability distribution $P_{w\sigma}(x)$. Therefore, $Q_k(\xi)$ is given as the k-fold convolution of $P_{w\sigma}(x)$:

$$Q_k(\xi)=\underbrace{P_{w\sigma}*P_{w\sigma}*\cdots*P_{w\sigma}(\xi)}_{k\ \text{times}}, \qquad \text{(A3)}$$

where $*$ is the convolution operator. Taking the Fourier transform of the above equation we get

$$\hat{Q}_k(\lambda)=[\hat{P}_{w\sigma}(\lambda)]^k \qquad \text{(A4)}$$

$$=[\phi\hat{P}_w(\lambda)+(1-\phi)\hat{P}_w(-\lambda)]^k. \qquad \text{(A5)}$$

Denoting as $\hat{g}(\lambda)$ and $\hat{h}(\lambda)$ the real and imaginary parts of $\hat{P}_w(\lambda)$, respectively, the last result becomes

$$\hat{Q}_k(\lambda,\Psi)=[\hat{g}(\lambda)+i\Psi\hat{h}(\lambda)]^k, \qquad \text{(A6)}$$

where in the last equation we have explicitly written that $\hat{Q}_k$ depends on the order parameter $\Psi=2\phi-1$. Inverse Fourier transforming the last equation and inserting the result into Eq. (A1) we get

$$P^{+}(\Psi,k)=\frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty}\hat{Q}_k(\lambda,\Psi)e^{-i\lambda\xi}d\lambda\, d\xi \qquad \text{(A7)}$$

$$=\frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty}[\hat{g}(\lambda)+i\Psi\hat{h}(\lambda)]^k e^{-i\lambda\xi}d\lambda\, d\xi. \qquad \text{(A8)}$$

Using the binomial theorem to expand $[\hat{g}(\lambda)+i\Psi\hat{h}(\lambda)]^k$, the last equation transforms into

$$P^{+}(\Psi,k)=\sum_{m=0}^{k} a(k,m)\Psi^m, \qquad \text{(A9)}$$

where the coefficients $a(k,m)$ are given by

$$a(k,m)=\frac{i^m}{2\pi}\binom{k}{m}\times\int_0^{\infty}\int_{-\infty}^{\infty}[\hat{g}(\lambda)]^{k-m}[\hat{h}(\lambda)]^m e^{-i\lambda\xi}d\lambda\, d\xi. \qquad \text{(A10)}$$

It is easy to show that $a(k,0)=1/2$ for any k and any $P_w(w)$. In order to do so, note that $\hat{g}(\lambda)$ is a symmetric function and $\hat{g}(0)=1$. Denoting as $G_k(\xi)$ the inverse Fourier transform of $[\hat{g}(\lambda)]^k$, it follows that $G_k(\xi)$ is also a symmetric function which satisfies $\int_{-\infty}^{\infty}G_k(\xi)d\xi=1$. From the above and Eq. (A10) it follows that

$$a(k,0)=\int_0^{\infty}G_k(\xi)d\xi=1/2. \qquad \text{(A11)}$$

On the other hand, for $m>0$ we can change the order of integration in Eq. (A10) and perform the integral over ξ by multiplying the integrand by $e^{-\delta\xi}$ and then taking the limit $\delta\to 0$:

$$a(k,m)=\frac{i^m}{2\pi}\binom{k}{m}\int_{-\infty}^{\infty}d\lambda[\hat{g}(\lambda)]^{k-m}[\hat{h}(\lambda)]^m \times\lim_{\delta\to 0}\int_0^{\infty}d\xi\, e^{-(\delta+i\lambda)\xi}$$

$$=\frac{-i^{m+1}}{2\pi}\binom{k}{m}\int_{-\infty}^{\infty}[\hat{g}(\lambda)]^{k-m}[\hat{h}(\lambda)]^m\frac{d\lambda}{\lambda}. \qquad \text{(A12)}$$

Equations (A11) and (A12) are the results (8) and (9) of the main text.

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[26] The reason for using $P(k)=k^{-(\gamma-1)}-(k+1)^{-(\gamma-1)}$, rather than the standard power law $P(k)=Ck^{-\gamma}$, is that the first is easier to generate numerically than the latter. Nevertheless, as we show below, the nature of the phase transition depends only on the asymptotic behavior of $P(k)$ for large k, which in both cases is $P(k)\sim k^{-\gamma}$.